Lec 12

A cocktail of vector space: Linear Transformation

My goal in this lecture is to introduce vector spaces of polynomials, vector spaces of matrices and finally vector spaces of sequences of real numbers. The last of the three vector spaces are not finite dimensional.

I Vector space of polynomials $P_n(t)$.

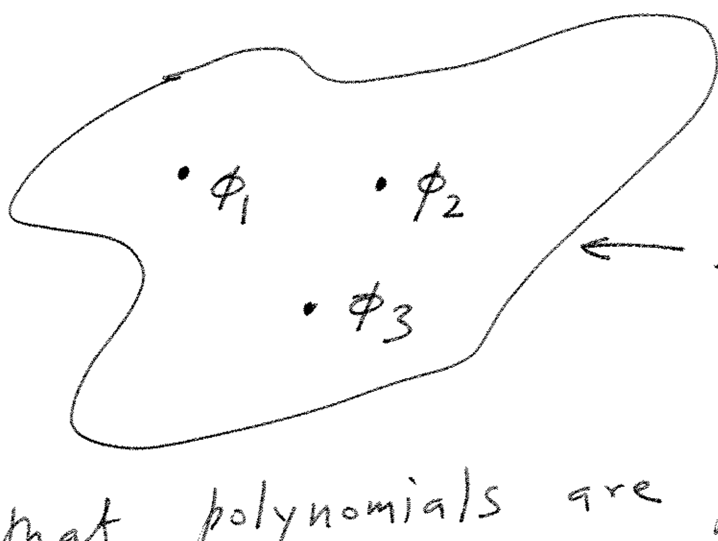
Consider the set of all polynomials of degree $\leq n$, in the variable t. We denote this set by $P_n(t)$.

Elements of $P_5(t)$ are

$\phi_1(t) = t^3 + 3t^2 + 5$

$\phi_2(t) = t^5 + t^4 + t$

$\phi_3(t)\ \ t^2 + t + 1$ etc.



$\leftarrow P_5(t)$.

Note that polynomials are points in $P_5(t)$.

Fact: $P_n(t)$ is a vector space.

Why: ① Because I can add two polynomials in $P_n(t)$.

Ex: $\phi_1(t)+\phi_2(t) = (t^3+3t^2+5) + (t^5+t^4+t)$

$$= \underbrace{t^3+t^4+t^3+3t^2+t+5}_{\in P_n(t)}.$$

2. Because I can scalar multiply a polynomial with a scalar.

Ex $\alpha \cdot \phi_1(t) = \alpha \cdot (t^3+3t^2+5)$.

$$= \alpha t^3 + 3\alpha t^2 + 5\alpha.$$

Addition & scalar multiplication satisfies all the different properties of vector space addition & scalar multiplication

(See page 359 in the text book).

Note that every polynomial in $P_n(t)$ can be written as a linear combination of the following polynomials in $P_n(t)$:—

$$\{1, t, t^2, t^3, t^4, t^5\}.$$

$$\therefore P_n(t) = \text{span}[1, t, t^2, t^3, t^4, t^5].$$

Moreover the polynomials $1, t, t^2, t^3, t^4, t^5$ are l.i. because if

$$\alpha_0 1 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3 + \alpha_4 t^4 + \alpha_5 t^5 = 0$$

The zero polynomial.

$$\Rightarrow \alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0.$$

Remark: We are not finding roots of a 5th order polynomial. We are actually setting a 5th order polynomial to a zero polynomial.

Hence, a basis of $P_n(t)$ is given by.

$$\{1, t, t^2, t^3, t^4, t^5\}.$$

Thus $\dim P_n(t) = 6$.

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II Vector space of matrices $M_{n\times m}$.

Consider the set of all $n\times m$ matrices.

(a) Elements of $M_{2\times 3}$ are

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

etc.

(b) Elements of $M_{2\times 2}$ are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

etc.

$M_{n\times m}$ is a vector space. because

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{pmatrix} =$$

$$\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1m}+b_{1m} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2m}+b_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1}+b_{n1} & a_{n2}+b_{n2} & \cdots & a_{nm}+b_{nm} \end{pmatrix}$$

$$\alpha \cdot \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} = \begin{pmatrix} \alpha a_{11} & \alpha a_{12} & \cdots & \alpha a_{1m} \\ \alpha a_{21} & \alpha a_{22} & \cdots & \alpha a_{2m} \\ \vdots & \vdots & & \vdots \\ \alpha a_{n1} & \alpha a_{n2} & \cdots & \alpha a_{nm} \end{pmatrix}$$

Addition and scalar multiplication satisfies all the different properties of vector space addition & scalar multiplication.

$M_{2\times 2}$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} =$$

$$a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

forms a basis of $M_{2\times 2}$.

Note! check independence.

$\dim M_{2\times 2} = 4$.

Example 1:

Let $S_{2\times2}$ be the set of all symmetric 2×2 matrices. Elements of $S_{2\times2}$ are of the form.

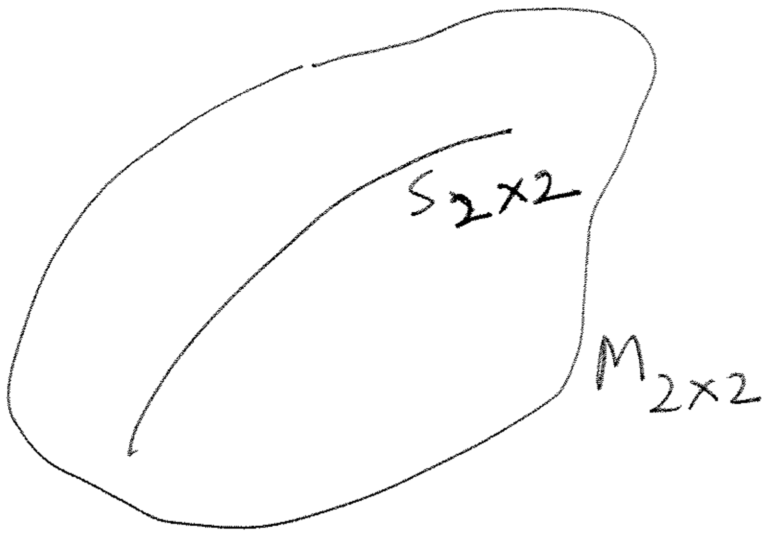
$$\begin{pmatrix} a & c \\ c & b \end{pmatrix}$$

$$= a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

forms a basis of $S_{2\times2}$.

$\dim S_{2\times2} = 3$.



$S_{2\times2} \subset M_{2\times2}$.

$S_{2\times2}$ is a 3 dimensional subspace of a 4 dimensional $M_{2\times2}$.

Example 2

Let $SK_{2\times2}$ be the set of all skew symmetric 2×2 matrices. Elements of $SK_{2\times2}$ are of the form.

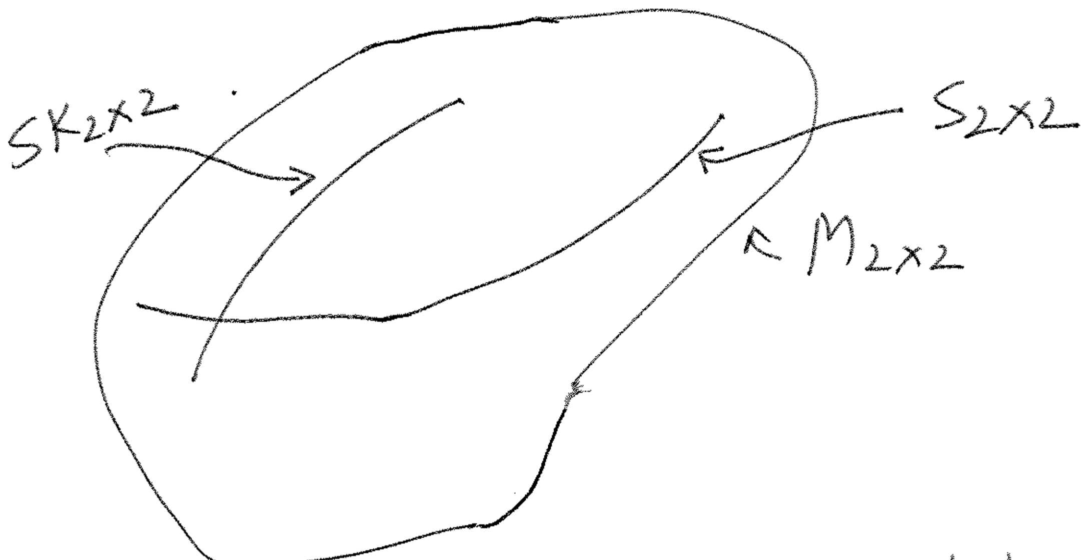
$$\begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix} = c\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(12.10)

$$\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

forms a basis of $SK_{2\times 2}$.

$\dim SK_{2\times 2} = 1$.



$SK_{2\times 2}$ & $S_{2\times 2}$ are two subspaces of

$M_{2\times 2}$.

The two subspaces intersect only at

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, the origin.

<u>Example 3</u>

Let us look at $M_{3\times3}$, $S_{3\times3}$ & $SK_{3\times3}$.

Check at $M_{3\times3}$ is 9 dimensional with basis given by

$$\left\{\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}, \cdots\cdots \begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}\right\}$$

$S_{3\times3}$ has elements of the form.

$$\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{12}&a_{22}&a_{23}\\a_{13}&a_{23}&a_{33}\end{pmatrix} = a_{11}\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix} + a_{12}\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}$$

$$+ a_{13}\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix} + a_{22}\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}$$

$$+ a_{23}\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix} + a_{33}\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}$$

$S_{3\times3}$ has basis given by

$$\left\{\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}\right\}.$$

$\dim S_{3\times3} = 6$.

Likewise check that $SK_{3\times3}$ is 3 dimensional. where

$$\begin{pmatrix}0&a_{12}&a_{13}\\-a_{12}&0&a_{23}\\-a_{13}&-a_{23}&0\end{pmatrix} = a_{12}\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}+a_{13}\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix}$$

$$+a_{23}\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix}.$$

$SK_{3\times 3}$ has basis given by

$$\left\{ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\}$$

$\dim SK_{3\times 3} = 3$.

Some Infinite Dimensional Vector Spaces.

Not all vector spaces we encounter in our life are finite dimensional. We will be describing two, most important, of those vector spaces, in this lecture and in the next.

I Vector space of sequences of real numbers.

Consider a vector space V whose elements are of the form

$$x = (x_1, x_2, x_3, x_4, \cdots\cdots) ; \ x_j \in \mathbb{R}.$$

x is an infinite sequence of real numbers.

Two elements x & y in V can be added point wise i.e if

$$y = (y_1, y_2, y_3, y_4, \cdots\cdots)$$

then we define

$$x + y = (x_1 + y_1, x_2 + y_2, x_3 + y_3, \cdots\cdots)$$

An element x in V can be scalar multiplied with a scalar α as follows.

$$\alpha . x = (\alpha x_1, \alpha x_2, \alpha x_3, \alpha x_4, \dots)$$

Addition and scalar multiplication satisfy all properties of a vector space.

— x —

Any element x in V can be written as

$$\begin{aligned} x = \; & x_1 . (1 \; 0 \; 0 \; 0 \; \dots) \\ & + x_2 . (0 \; 1 \; 0 \; 0 \; \dots) \\ & + \dots \\ & + \dots \\ & + x_j . (0, 0, \dots, 0, \underset{j^{th} \text{ spot}}{1}, 0, \dots) \\ & + \dots \\ & + \dots \end{aligned}$$

Thus a basis in V is given by

$$\{(1,0,\dots\dots),(0,1,0,\dots\dots),(0,0,1,0,\dots\dots),\\ \dots\dots,(0,0,\dots,0,1,0,\dots\dots),\dots\dots\}$$

There are infinitely many elements in V making V an infinite dimensional vector space.

—— x ——

Examples of elements in V

a. $(1,2,3,4,\dots\dots)$

b. $(1,1,1,1,\dots\dots)$

c $(1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\dots\dots)$.

d $(1,\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},\dots\dots)$.

e. $(h, ah, a^2h, a^3h, a^4h, \ldots\ldots)$

$$a, h \in \mathbb{R}.$$

f. $(h_1, h_2, h_3, h_4, \ldots\ldots, h_j, \ldots\ldots)$

$$h_j = -\alpha_1 h_{j-1} - \alpha_2 h_{j-2}, \; j = 3, 4, \ldots\ldots$$

$$h_1, h_2, \alpha_1, \alpha_2 \in \mathbb{R}.$$

Remark:

d and e are called geometric sequence.

Elements in f are generated recursively.

c is famous because

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \cdots$$

is not finite and is an example of a sequence whose elements are becoming smaller but the sum is not.

There are many interesting subsets of V that are also vector spaces.

1. $U \triangleq \{ x \in V : x \text{ is bounded} \}$.

Note that a sequence

$(x_1, x_2, x_3, \dots)$

is bounded if $\exists\, M : |x_j| \leq M \ \forall j$.

Remark: The sequence ⓐ on page 12·16 is not bounded, ⓑ, ⓒ, ⓓ are bounded.

If x, y are two bounded sequences in V, then $x+y$ is also bounded

αx is also " for some α.

Proof: $\exists\, M : |x_j| \leq M \ \forall j$

$\exists\, N : |y_j| \leq N \ \forall j$.

$\Rightarrow |x_j + y_j| \leq |x_j| + |y_j| \leq M + N \ \forall j$.

Hence $x+y$ is bounded.

$|\alpha x_j| \leq |\alpha| M \ \forall j$

Hence αx is also bounded

Hence U itself is a vector space.

— x —

2. $W \triangleq \left\{ x \in V : \sum_{j=1}^{\infty} |x_j| \text{ is bounded} \right\}$.

W are the set of all sequences in V that are absolutely summable.

Note that

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{1 - 1/2} = 2$$

Hence the sequence (d) on page 12-16 is in W but, (a), (b), (c) are not.

— x —

If $|a| < 1$, the sequence (e) on page 12·17 is also in W. This is because

$$h + ah + a^2h + \cdots$$

$$\leq |h| \left[1 + |a| + |a|^2 + |a|^3 + \cdots \right]$$

$$= |h| \frac{1}{1-|a|} \quad \text{if } |a| < 1$$

— x —

It can be shown (but we dont show it here) that the sequence (f) on page 12·17 is in W if the polynomial

$$\lambda^2 + \alpha_1 \lambda + \alpha_2$$

has roots with magnitude < 1.

3. Continuing the story of sequences a bit further we define

$$\ell^2 \triangleq \{ x \in V : \sum_{j=1}^{\infty} |x_j|^2 \text{ is bounded} \}.$$

It is easily seen that every element of W is also in ℓ^2, ie every summable sequence is also square summable.

$$W \subset \ell^2.$$

The sequence (c) on page 12·16 is not in W but it is actually in ℓ^2 ie.

$$1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{4}\right)^2 + \cdots$$

is finite.

Define

$$\ell^p \triangleq \left\{ x \in V : \sum_{j=1}^{\infty} |x_j|^p \text{ is bounded} \right\}$$

$$p = 1, 2, \ldots$$

The subspace W that we have defined on page 12·19 is ℓ^1. In general we can show that

$$\ell^1 \subset \ell^2 \subset \ell^3 \subset \cdots \subset V$$

We will be only interested in ℓ^1 and ℓ^2 for now.

Examples of some sequence generators :—

Example :

Let A be a 2×2 matrix with eigenvalues at $1/2, -1/2$. characteristic polynomial of A is

$$(\lambda - \tfrac{1}{2})(\lambda + \tfrac{1}{2}) = \lambda^2 - \tfrac{1}{4}.$$

$$A = \begin{pmatrix} 0 & 1 \\ \frac{1}{4} & 0 \end{pmatrix} \leftarrow \text{companion form.}$$

Let b and c be two vectors given by

$$b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad c = (3 \quad 4)$$

Define a sequence

$$(h_1, h_2, h_3, h_4, \ldots\ldots)$$

as follows

$$h_j = c A^{j-1} b$$

ie

$$(cb, cAb, cA^2b, cA^3b, \ldots\ldots)$$

is the sequence.

using matlab, the sequence is written as follows in page 12.25a

Q: Can we add the sequence

$cb + cAb + cA^2b + cA^3b + \cdots$

and get a finite number.

Ans: Of course

$cb + cAb + cA^2b + \cdots$

$= c(I + A + A^2 + A^3 + \cdots)b$

So we need to know if

$S = I + A + A^2 + A^3 + \cdots$

exists. writing

$f(A) = I + A + A^2 + A^3 + \cdots = \alpha_0 I + \alpha_1 A$

Cayley Hamilton again.

We compute α_0 and α_1 by our well known trick — "Replace A by eigenvalue of A."

We obtain.

$$\alpha_0 + \alpha_1 \frac{1}{2} = f\left(\frac{1}{2}\right)$$
$$\alpha_0 - \alpha_1 \frac{1}{2} = f\left(-\frac{1}{2}\right)$$

$$f(t) = 1 + t + t^2 + t^3 + \cdots\cdots = \frac{1}{1-t}$$

if $|t| < 1$.

$$\therefore f\left(\tfrac{1}{2}\right) = \frac{1}{1-\frac{1}{2}} = 2.$$
$$f\left(-\tfrac{1}{2}\right) = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}.$$

Thus we get

$$\begin{pmatrix} 1 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2/3 \end{pmatrix}$$

$$\alpha_0 = \frac{\begin{vmatrix} 2 & 1/2 \\ 2/3 & -1/2 \end{vmatrix}}{\begin{vmatrix} 1 & 1/2 \\ 1 & -1/2 \end{vmatrix}} ; \quad \alpha_1 = \frac{\begin{vmatrix} 1 & 2 \\ 1 & 2/3 \end{vmatrix}}{\begin{vmatrix} 1 & 1/2 \\ 1 & -1/2 \end{vmatrix}}$$

$$\alpha_0 = \frac{4}{3}$$
$$\alpha_1 = 4/3$$

(12.25a)

cb $\quad cAb$ $\quad cA^2b$ $\quad cA^9b$ $\quad cA^5b$

11.000	7.0000	2.7500	1.7500	0.6875	0.4375
0.1719	0.1094	0.0430	0.0273	0.0107	0.0068
0.0027	0.0017	0.0007	0.0004	0.0002	0.0001
0.0000	0.0000				

$cA^{16}b$ $\quad cA^{17}b$

$cA^jb \approx 0$ $j \geq 18$, upto 4^{th} place of decimal.

$$(I - A)^{-1} = \begin{pmatrix} \frac{4}{3} & \frac{4}{3} \\ \frac{1}{3} & \frac{4}{3} \end{pmatrix}$$

$$c(I-A)^{-1}b = 24$$

It follows that

$$cb + cAb + cA^2b + \cdots$$

$$= c\left[\alpha_0 I + \alpha_1 A\right] b$$

$$= \alpha_0 (cb) + \alpha_1 (cAb)$$

<u>Remark:</u>

One can show that

$$I + A + A^2 + A^3 + \cdots = (I - A)^{-1}$$

if all the eigenvalues of A have magnitude < 1.

Hence

$$cb + cAb + \cdots = c(I-A)^{-1}b$$

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<u>True Fact One:</u>

The sequence $(h_1, h_2, \ldots)$ on page 12.23 belongs to ℓ^1, ie the sequence is absolutely summable, if all the eigenvalues of A have magnitude < 1.

Why do we care about this sequence?

Discrete Time Recursion

$$\underline{x}(k+1) = A\,\underline{x}(k) + b\,u(k).$$
$$y(k) = c\,\underline{x}(k)$$

A is a $n \times n$ matrix

b is a $n \times 1$ vector

c is a $1 \times n$ vector.

$u = (u_1, u_2, u_3, \cdots\cdots)$

$\underline{x} = (\underline{x}_1, \underline{x}_2, \underline{x}_3, \cdots\cdots)$ ← sequence of $n \times 1$ vectors.

$y = (y_1, y_2, y_3, \cdots\cdots)$

We assume $\underline{x}_1 = 0$ ← Zero initial condition.

It follows that

$$\underline{x}_2 = A\underline{x}_1 + bu_1 = bu_1$$
$$\underline{x}_3 = A\underline{x}_2 + bu_2 = Abu_1 + bu_2$$

$$\underline{X}_4 = A^2 b u_1 + A b u_2 + b u_3$$

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Thus. $y_1 = 0$ and

$$y_2 = c\underline{X}_2 = c b u_1$$

$$y_3 = c\underline{X}_3 = cAbu_1 + cbu_2$$

$$y_4 = c\underline{X}_4 = cA^2bu_1 + cAbu_2 + cbu_3$$

$$y_{j+2} = c\underline{X}_{j+2} = cA^j b u_1 + cA^{j-1} b u_2 + \cdots + cbu_{j+1}$$

$j = 0, 1, 2, \ldots\ldots$

Q: When is it true that the sequence y is bounded assuming that the sequence u is bounded.

Q. When is it true that the sequence $y \in \ell^1$ (i.e absolutely summable) assuming that $u \in \ell^1$ (i.e u is absolutely summable).

In order that the sequence y is bounded it follows that $\exists M$:

$$|y_j| \le M \quad \forall j = 2, 3, 4, \dots$$

If we assume that u is bounded it follows that $\exists N : |u_j| \le N \quad \forall j$.

We have

$$|y_{j+2}| \le \left[|CA^j b| + |CA^{j-1} b| + \cdots + |Cb| \right] N$$

$$\forall j$$

A sufficient condition for y to be bounded is that the sequence

$$(Cb, CAb, CA^2b \ldots\ldots)$$

is absolutely summable. Thus $\exists\, R > 0$:

$$|Cb| + |CAb| + |CA^2b| + \cdots \leq R$$

implying that

$$|y_{j+2}| \leq RN \quad \forall j.$$

Choosing $M = RN$ the result follows. Finally one concludes from page 12·26 that if all the eigenvalues of A have magnitude < 1, the sequence $(Cb, CAb, \ldots\ldots)$ is absolutely summable and the sequence y would be bounded as long as the sequence u is bounded.

Thus we write the following definition:

Def (BIBO stability):

A discrete time recursion on page 12·27 is called Bounded Input Bounded output stable if every bounded u produces a bounded y assuming zero initial condition.

Fact: The discrete time recursion on page 12·27 is BIBO stable if all eigenvalues of A have magnitude < 1.

Remark: Under a suitable additional restriction on the matrix A, b, c, the eigenvalue condition is also necessary for BIBO stability.

To get additional insight, assume that one eigenvalue of A has magnitude ≥ 1. We want to know under what condition is the discrete time recursion on page 12·27 BIBO unstable, i.e. y would be unbounded for a bounded u.

Assume for the sake of argument that A is 3×3, b is 3×1, c is 1×3 and A has distinct eigenvalues at $\lambda_1, \lambda_2, \lambda_3$ and that $|\lambda_1| > 1$. Let us choose a bounded u given by $u_j = 1, j = 1, 2, 3, \ldots$.

$$u = (1 \quad 1 \quad 1 \quad 1 \quad \ldots)$$

It follows from page 12·28 that

$$y_{j+2} = cb + cAb + cA^2b + \cdots + cA^jb.$$

Using Cayley Hamilton we write

$$A^3 = \alpha_0 I + \alpha_1 A + \alpha_2 A^2$$

$$\Rightarrow c A^j b = \bar{\alpha}_0\, cb + \bar{\alpha}_1\, cAb + \bar{\alpha}_2\, cA^2 b$$

for some choice of j dependent $\bar{\alpha}_0, \bar{\alpha}_1, \bar{\alpha}_2$.

$\bar{\alpha}_0, \bar{\alpha}_1, \bar{\alpha}_2$ are computed by writing

$$A^j = \bar{\alpha}_0 I + \bar{\alpha}_1 A + \bar{\alpha}_2 A^2$$

$$\Rightarrow \begin{pmatrix} \lambda_1^j \\ \lambda_2^j \\ \lambda_3^j \end{pmatrix} = \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 1 & \lambda_2 & \lambda_2^2 \\ 1 & \lambda_3 & \lambda_3^2 \end{pmatrix} \begin{pmatrix} \bar{\alpha}_0 \\ \bar{\alpha}_1 \\ \bar{\alpha}_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \bar{\alpha}_0 \\ \bar{\alpha}_1 \\ \bar{\alpha}_2 \end{pmatrix} = \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 1 & \lambda_2 & \lambda_2^2 \\ 1 & \lambda_3 & \lambda_3^2 \end{pmatrix}^{-1} \begin{pmatrix} \lambda_1^j \\ \lambda_2^j \\ \lambda_3^j \end{pmatrix}$$

Hence we obtain

$$cA^j b =$$

$$\begin{pmatrix} cb & cAb & cA^2b \end{pmatrix} \begin{pmatrix} \bar{\alpha}_0 \\ \bar{\alpha}_1 \\ \bar{\alpha}_2 \end{pmatrix}$$

$$= \begin{pmatrix} cb & cAb & cA^2b \end{pmatrix} \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 1 & \lambda_2 & \lambda_2^2 \\ 1 & \lambda_3 & \lambda_3^2 \end{pmatrix}^{-1} \begin{pmatrix} \lambda_1^j \\ \lambda_2^j \\ \lambda_3^j \end{pmatrix}$$

$$= \begin{pmatrix} \theta_1 & \theta_2 & \theta_3 \end{pmatrix}$$

writing

$$cA^j b = \begin{pmatrix} \theta_1 & \theta_2 & \theta_3 \end{pmatrix} \begin{pmatrix} \lambda_1^j \\ \lambda_2^j \\ \lambda_3^j \end{pmatrix}$$

We obtain from page 12·32

$$y_{j+2} = (\theta_1 \quad \theta_2 \quad \theta_3) \begin{pmatrix} 1 + \lambda_1 + \lambda_1^2 + \cdots \\ 1 + \lambda_2 + \lambda_2^2 + \cdots \\ 1 + \lambda_3 + \lambda_3^2 + \cdots \end{pmatrix}$$

If $|\lambda_1| > 1$, then y_{j+2} is unbounded provided $\theta_1 \neq 0$.

This is the additional restriction on C, A, B.

```
>> C=[0 1 1]

C =

     0     1     1

>> B=[1 ; 1 ; 1]

B =

     1
     1
     1

>> A=[2 0 0;0 .5 0;0 0 .25]

A =

    2.0000         0         0
         0    0.5000         0
         0         0    0.2500

>> v=[C*B C*A*B C*A^2*B]

v =

    2.0000    0.7500    0.3125

>> M=[1 2 4;1 .5 .25;1 .25 .25*.25]

M =

    1.0000    2.0000    4.0000
    1.0000    0.5000    0.2500
    1.0000    0.2500    0.0625

>> N=inv(M)

N =

    0.0476   -1.3333    2.2857
   -0.2857    6.0000   -5.7143
    0.3810   -2.6667    2.2857

>> theta=v*N

theta =

   -0.0000    1.0000    1.0000

>> C=[1 2 3]

C =
```

Example:

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$c = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 1 & \lambda_2 & \lambda_2^2 \\ 1 & \lambda_3 & \lambda_3^2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 1 & \lambda_2 & \lambda_2^2 \\ 1 & \lambda_3 & \lambda_3^2 \end{pmatrix}^{-1}$$

This mode does not show up in the output.

$$cA^j b = 0(2)^j + 1\left(\tfrac{1}{2}\right)^j + 1\left(\tfrac{1}{4}\right)^j$$

In this example, $\theta_1 = 0$. Hence y is bounded even though A has an eigenvalue at $\lambda = 2$.

```
     1     2     3

>> v=[C*B C*A*B C*A^2*B]

v =

    6.0000    3.7500    4.6875

>> theta=v*N

theta =

    1.0000    2.0000    3.0000

>> P=[1 2 1;2 1 3;1  1 1]

P =

     1     2     1
     2     1     3
     1     1     1

>> det(P)

ans =

     1

>> Q=inv(P)

Q =

   -2.0000   -1.0000    5.0000
    1.0000   -0.0000   -1.0000
    1.0000    1.0000   -3.0000

>> C1=C*P

C1 =

     8     7    10

>> A1=Q*A*P

A1 =

   -3.7500   -7.2500   -4.2500
    1.7500    3.7500    1.7500
    2.2500    3.7500    2.7500

>> B1=Q*B

B1 =

     2
     0
    -1
```

Example:

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & \tfrac{1}{2} & 0 \\ 0 & 0 & \tfrac{1}{4} \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$c = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$$

$$cA^j b = 1(2)^j + 2\left(\tfrac{1}{2}\right)^j + 3\left(\tfrac{1}{4}\right)^j$$

Because of this term, y is not bounded.

Example

$$A = \begin{pmatrix} -3\frac{3}{4} & -7\frac{1}{4} & -4\frac{1}{4} \\ 1\frac{3}{4} & 3\frac{3}{4} & 1\frac{3}{4} \\ 2\frac{1}{4} & 3\frac{3}{4} & 2\frac{3}{4} \end{pmatrix}$$

$$b = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \qquad c = \begin{pmatrix} 8 & 7 & 10 \end{pmatrix}$$

```
>> v1=[C1*B1 C1*A1*B1 C1*A1^2*B1]

v1 =

    6.0000    3.7500    4.6875

>> theta1=v1*N

theta1 =

    1.0000    2.0000    3.0000

>>
```

$$CA^j b = 1\,2^j + 2\left(\tfrac{1}{2}\right)^j + 3\left(\tfrac{1}{4}\right)^j$$

Same as the previous example

In summary, when all the eigenvalues of the A matrix show up in the output, a necessary and sufficient condition for BIBO stability of the discrete time recursion on page 12·27 is that

all eigenvalues of A have magnitude < 1.

Remark:

From page 12.28, we can also do the following calculation. Recall that

$$y_1 = 0$$
$$y_2 = cbu_1$$
$$y_3 = cAbu_1 + cbu_2$$
$$y_4 = cA^2bu_1 + cAbu_2 + cbu_3$$

It follows that

$$|y_1| + |y_2| + |y_3| + \cdots$$
$$\leq \left[|cb||u_1|\right] + \left[|cAb||u_1| + |cb|u_2\right] + \left[|cA^2b||u_1| + |cAb||u_2| + |cb||u_3|\right] + \cdots$$
$$= \left[|cb| + |cAb| + |cA^2b| + \cdots\right]\left[|u_1| + |u_2| + |u_3| + \cdots\right]$$

"If $|cb| + |cAb| + |cA^2b| + \cdots \leq M$

it would follow that an absolutely summable u would produce an absolutely summable y."

answering the question raised in page 12·29.

Conversely, if $(cb, cAb, \ldots)$ is not absolutely summable then $\exists$ a sequence

$u = (1, 0, 0, \ldots)$

such that

$y = (cb, cAb, \ldots)$ is not absolutely summable. Hence:

Absolutely summable u produces absolutely summable y iff $(cb, cAb, \ldots)$ is abs. summable

Example :

Consider $P_n(t)$, the set of all polynomials of degree $\leq n$. Choose $n=5$. Define S as follows:

$$S \triangleq \{ p(t) \in P_n(t) : t=1 \text{ is a root of } p(t) \}_{n=5}.$$

Find a basis of S.

Sol^n: Since $t=1$ is a root of $p(t)$, it must be of the form

$$(t-1)\,\phi(t)$$

where $\phi(t)$ is a polynomial of degree ≤ 4.

ie $\phi(t) = a_4 t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0$

It follows that

$$p(t) = a_4(t-1)t^4 + a_3(t-1)t^3 + a_2(t-1)t^2 + a_1(t-1)t + a_0(t-1).$$

Basis of S is given by

$$\{(t-1)t^4, (t-1)t^3, (t-1)t^2, (t-1)t, t-1\}.$$

Example:

Repeat previous example defining S as follows:

$$S \triangleq \{ p(t) \in P_n(t) : i \,\&\, -i \text{ are roots of } p(t),\ n=5 \}.$$

Find a basis of S.

Solution:

Since $i, -i$ are roots of $p(t)$, t^2+1 is a factor of $p(t)$. It follows that $p(t)$ must be of the form

$$(t^2+1)\,\phi(t)$$

where $\phi(t)$ is a polynomial of degree ≤ 3 i.e.

$$\phi(t) = a_3 t^3 + a_2 t^2 + a_1 t + a_0 .$$

It follows that

$$p(t) = a_3 (t^2+1) t^3 + a_2 (t^2+1) t^2 + a_1 (t^2+1) t + a_0 (t^2+1).$$

Basis of S is given by

$$\{ (t^2+1) t^3, (t^2+1) t^2, (t^2+1) t, (t^2+1) \}$$

Example:

Let A be a square $n \times n$ matrix. We can define two matrices from A as follows:

$$L = \frac{1}{2}(A + A^T)$$

$$M = \frac{1}{2}(A - A^T)$$

where $L + M = A$. It is easy to see that L is symmetric whereas M is skewsymmetric.

"Every $n \times n$ matrix can be written as a sum of symmetric and skew symmetric matrix."

Actually one can show that this decomposition

"$A = L + M$"

is unique.

A set of $n \times n$ matrices

A set of $n \times n$ symmetric matrices

$$M_{n\times n} = S_{n\times n} \oplus SK_{n\times n}$$

A set of $n \times n$ skew symmetric matrices.

Fibonacci Sequence & Golden Search :–

A very important sequence of integers goes by the name Fibonacci-Sequence. It is generated using the recurrence

$$h_{j+2} = h_{j+1} + h_j \ , \ j = 0, 1, 2, 3, \ldots$$

$$h_0 = 1, \ h_1 = 1$$

The sequence looks like

$$(\underset{h_0}{1}, \underset{h_1}{1}, \underset{h_2}{2}, 3, 5, 8, 13, 21, 34, 55, 89, \ldots)$$

Q: Calculate h_{100}?

In order to write an expression for the 100^{th} element, h_{100} we write

$x_1(j) = h_j$ ← state variables.

$x_2(j) = h_{j+1}$

$$x_1(j+1) = h_{j+1} = x_2(j)$$

$$x_2(j+1) = h_{j+2} = h_{j+1} + h_j$$

$$= x_2(j) + x_1(j)$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(j+1) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(j)$$

$$h_j = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(j)$$

Denoting

$$C = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \qquad \underline{X}(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$h_j = C A^j \underline{X}(0)$$

Hence

$$h_{100} = C A^{100} \underline{X}(0).$$

We need to calculate A^{100}.

Writing

$$A^N = \alpha_0 I + \alpha_1 A$$

Char. poly of A is

$$\det\begin{pmatrix} \lambda & -1 \\ -1 & \lambda-1 \end{pmatrix} = \lambda^2 - \lambda - 1$$

Roots are at

$$\lambda = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Let

$$\lambda_1 = (1+\sqrt{5})/2 \; ; \; \lambda_2 = (1-\sqrt{5})/2$$

$$\left.\begin{aligned} \lambda_1^N &= \alpha_0 + \alpha_1 \lambda_1 \\ \lambda_2^N &= \alpha_0 + \alpha_1 \lambda_2 \end{aligned}\right\} \quad \alpha_1 = \frac{\lambda_1^N - \lambda_2^N}{\lambda_1 - \lambda_2}$$

$$\alpha_0 = \frac{\lambda_1 \lambda_2^N - \lambda_2 \lambda_1^N}{\lambda_1 - \lambda_2}$$

(12.47)

$$A^N = \begin{pmatrix} \alpha_0 & 0 \\ 0 & \alpha_0 \end{pmatrix} + \begin{pmatrix} 0 & \alpha_1 \\ \alpha_1 & \alpha_1 \end{pmatrix} = \begin{pmatrix} \alpha_0 & \alpha_1 \\ \alpha_1 & \alpha_0+\alpha_1 \end{pmatrix}$$

$$\alpha_1 + \alpha_0 = \frac{\lambda_1^N - \lambda_2^N + \lambda_1\lambda_2^N - \lambda_2\lambda_1^N}{\lambda_1 - \lambda_2}$$

$$= \frac{\lambda_1^N(1-\lambda_2) - \lambda_2^N(1-\lambda_1)}{\lambda_1-\lambda_2}$$

$$A^N = \frac{\begin{pmatrix} \lambda_1\lambda_2^N - \lambda_2\lambda_1^N & \lambda_1^N - \lambda_2^N \\ \lambda_1^N - \lambda_2^N & (1-\lambda_2)\lambda_1^N - (1-\lambda_1)\lambda_2^N \end{pmatrix}}{\lambda_1 - \lambda_2}.$$

check Mat

$1-\lambda_1 = \lambda_2$, $1-\lambda_2 = \lambda_1$

$\lambda_1 - \lambda_2 = \sqrt{5}$, $-\lambda_1\lambda_2 = 1$

It follows that

$$A^N = \begin{pmatrix} \underbrace{-\lambda_1\lambda_2}_{=1}\left(\lambda_1^{N-1} - \lambda_2^{N-1}\right) & \lambda_1^N - \lambda_2^N \\ \lambda_1^N - \lambda_2^N & \lambda_1^{N+1} - \lambda_2^{N+1} \end{pmatrix} \Big/ \sqrt{5}.$$

Hence

$$A^N = \begin{pmatrix} \lambda_1^{N-1} - \lambda_2^{N-1} & \lambda_1^N - \lambda_2^N \\ \lambda_1^N - \lambda_2^N & \lambda_1^{N+1} - \lambda_2^{N+1} \end{pmatrix} \Big/ \sqrt{5}$$

$$\begin{aligned} h_N &= c\, A^N \underline{x}(0) \\ &= \left(\lambda_1^{N-1} - \lambda_2^{N-1} + \lambda_1^N - \lambda_2^N\right) \Big/ \sqrt{5} \\ &= \left[(\lambda_1+1)\,\lambda_1^{N-1} - (\lambda_2+1)\,\lambda_2^{N-1}\right] \Big/ \sqrt{5} \\ &= \frac{3+\sqrt{5}}{2\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{N-1} - \left(\frac{3-\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{N-1} \end{aligned}$$

$$h_{100} = \underbrace{\frac{3+\sqrt{5}}{2\sqrt{5}}}_{=a}\underbrace{\left(\frac{1+\sqrt{5}}{2}\right)}_{=c}^{99} - \underbrace{\frac{3-\sqrt{5}}{2\sqrt{5}}}_{=b}\underbrace{\left(\frac{1-\sqrt{5}}{2}\right)}_{=d}^{99}$$

This number is very small.

$a = 1{\cdot}1708$

$b = {\cdot}1708$

$c = 1{\cdot}6180$

$d = -{\cdot}6180$

$$h_{100} \approx \frac{3+\sqrt{5}}{2\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{99}$$

$$h_N = a\,c^{N-1} - b\,d^{N-1}$$

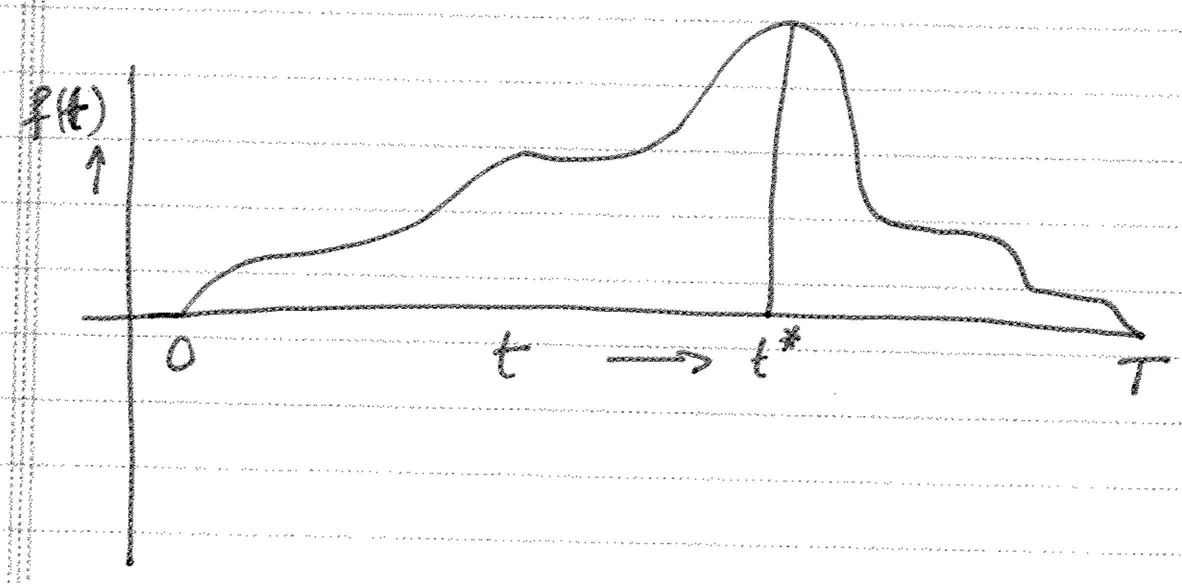
Matlab session to compute Fibbonaci Numbers

12·50

```
>> b=(3-sqrt(5))/(2*sqrt(5))

b =    0.1708

>> a=(3+sqrt(5))/(2*sqrt(5))

a =   1.1708

>> c=(1+sqrt(5))/2

c =   1.6180

>> d=(1-sqrt(5))/2

d =   -0.6180

>> h5=a*c^4-b*d^4

h5 = 8

>> h10=a*c^9-b*d^9

h10 =       89

>> a*c^9

ans =       88.9978

>> h15=round(a*c^14-b*d^14)

h15 =       987

>> round(a*c^14)

ans =       987
```

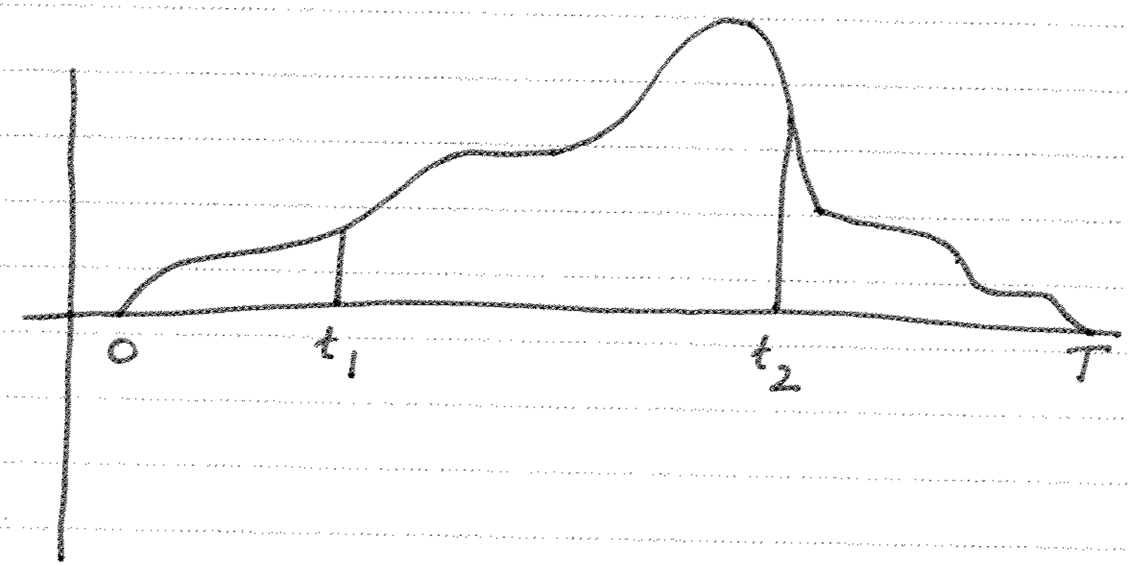
What would be an use of the Fibonacci Numbers that made Fibonacci famous :→

Fibonacci Numbers are useful in computing the maximum of an unimodal function by search method.



Let $f(t)$ be a function in the interval $[0, T]$ and assume that $f(t)$ has a maximum at t^* and that $f(t)$ decreases monotonously on both sides of t^*.

We dont know t^* and we would like to search for t^* as follows:



1. The initial region of uncertainty for t^* is $[0, T]$
2. Choose two points $t_1, t_2 : 0 < t_1 < t_2 < T$ and evaluate $f(t_1)$ & $f(t_2)$.
3. If $f(t_1) > f(t_2)$ choose the new region of uncertainty as $[0, t_2]$

   If $f(t_1) < f(t_2)$ choose the new region of uncertainty as $[t_1, T]$

12.53

4. Rename the new region of uncertainty as $[0, T]$ and Go back to step 1.

Remark:

Note that every iteration of the algorithm reduces the uncertainty by a certain fraction. f. Each iteration takes 2 evaluations of the function.

If we are allowed a total of $2m$ evaluations, then the region of uncertainty is reduced by a fraction f^m in m iterations.

Q: How do we choose the points t_1, t_2 so that f^m is smallest?

One strategy is to choose t_1 & t_2 symmetrically in every iteration and choose

$$t_1 = \Delta \ , \ t_2 = T - \Delta$$

where we assume $T/2 < \Delta < T$

In this case $f = (T-\Delta)/T = \left(1 - \frac{\Delta}{T}\right)$

In m iterations, there is a reduction of the region of uncertainty by

$$\left(1 - \frac{\Delta}{T}\right)^m$$

requiring $2m$ evaluations of the function.

By choosing Δ sufficiently close to $T/2$ (you cannot choose $\Delta = T/2$ because in that case $t_1 = t_2$) we have

$$\left(1 - \frac{\Delta}{T}\right)^m \approx \frac{1}{2^m},$$

The best achievable reduction of the region of uncertainty per evaluation of the function, using the search technique described on page 12:52 is

$F_{opt.} = \frac{1}{2^m}$ for every 2m evaluations of the function

# of eval.	2	4	6	8	10	12	14
F_{opt}	$1/2$	$1/4$	$1/8$	$1/16$	$1/32$	$1/64$	$1/128$

For n evaluations, the reduction ratio is $\left(\frac{1}{2}\right)^{n/2} = \left(\frac{1}{\sqrt{2}}\right)^n$

Golden Search :—

Golden search is a modification of the search algorithm in page 12·52 and shows that if f_n is the n^{th} Fibonacci number, it takes n evaluation to reduce the region of uncertainty to $2/f_{n+1}$, $n=2,3,\ldots$

# of Eval.	2	3	4	5	6	7	8
F_{opt}	$2/3$	$2/5$	$2/8$	$2/13$	$2/21$	$2/34$	$2/55$

For (n large) n evaluations, the reduction ratio :

$$\frac{2}{f_{n+1}} \approx \frac{4\sqrt{5}}{3+\sqrt{5}}\left(\frac{2}{1+\sqrt{5}}\right)^n$$

12:57

Golden Search Algorithm:

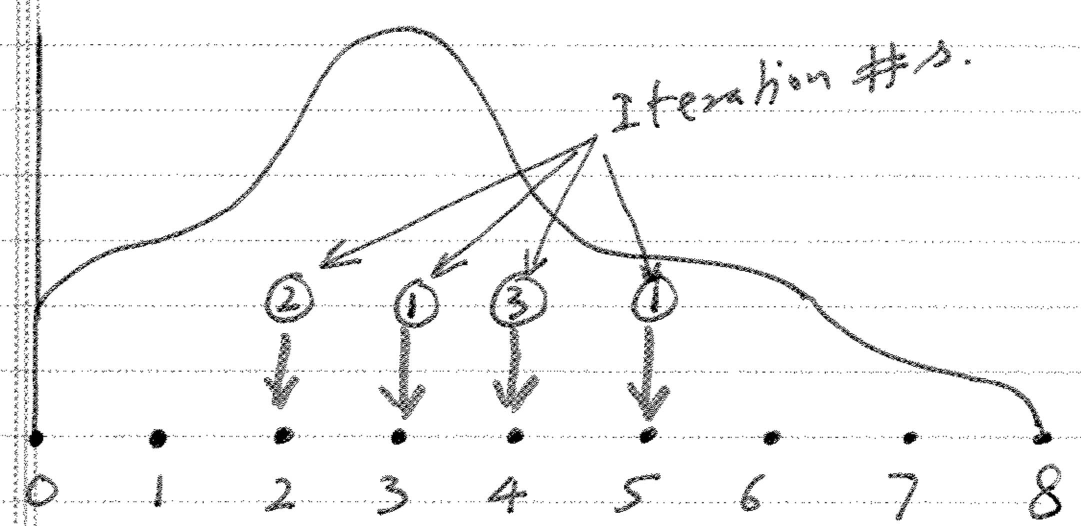
1. Choose n the # of evaluations.
2. Calculate the Fibbonaci numbers
   $f_0, f_1, f_2, \ldots, f_{n+1}$.
3. Divide the region $[0, T]$ into f_{n+1} equally spaced intervals of width
   $\omega = T/f_{n+1}$.
4. Choose $t_1 = f_{n-1}\,\omega = \dfrac{f_{n-1}}{f_{n+1}} T$
   $t_2 = f_n\,\omega = \dfrac{f_n}{f_{n+1}} T$
   Calculate $f(t_1), f(t_2)$.
5. If $f(t_1) > f(t_2)$ choose the new region of uncertainty as $[0, t_2]$

   If $f(t_1) < f(t_2)$ choose the new region of uncertainty as $[t_1, T]$
6. Rename the new region of uncertainty as $[0, T]$. Define $n = n-1$.
   If $n = 2$ Stop otherwise Go to Step 3.

A remarkable property of the Golden Algorithm is that only in the first iteration # evaluation in step 4 is 2. Subsequently, only one evaluation is sufficient and one can use the value of the function from the previous iteration.

Illustration of the Golden Search Algorithm:

Let $[0,T]=[0,8]$



① Let $n=5$.

② $f_0=1$ $f_1=1$ $f_2=2$ $f_3=3$ $f_4=5$
$f_5=8$

③ Divide $[0,8]$ into 8 region of width $w=1$ each.

④ Choose $t_1=3$ $t_2=5$
Calculate $f(3)$, $f(5)$.

⑤ Since $f(3)>f(5)$ the new region is $[0,5]$,

⑥ Define $n=4$.

(7) Divide $[0,5]$ into 5 regions of width $w=1$ each.

(8) Choose $t_1=2$, $t_2=3$
Calculate $f(2)$, $f(3)$

already calculated from prev. iteration.

(9) Since $f(3) > f(2)$ the new region is $[2,5]$

(10) Define $n=3$

(11) Divide $[2,5]$ into 3 regions of width $w=1$ each.

(12) Choose $t_1=3$, $t_2=4$
Calculate $f(3)$, $f(4)$

already calculated from previous iteration.

(13) Since $f(3) > f(4)$, the new region is $[2,4]$

(14) Define $n=2$ Stop.

Conclusion:

The maximum of the function f is in the interval $[2, 4]$ concluded after 4 evaluations ($n=4$). The reduction is $2/8$ which is $2/f_{n+1}$ as claimed earlier.

—— × ——